Levy Flights and continuous time random walks.

Start with discrete random walk in time

$y_n \equiv x_n - x_{n+1}$ is the stepsize.

For Brownian motion we have:

$$P(y_n) = \frac{1}{\sqrt{4\pi D t}} \cdot \exp\left(-\frac{y^2}{4Dt}\right)$$ ^ $\sigma = \sqrt{2Dt}$ RMS of the distribution

Thus the discrete distribution for: $n = t/T$ where T is the time increment is:

$$p(x,n) = \frac{1}{\sqrt{4\pi D(Tn)}} \cdot \exp\left(-\frac{x^2}{4D(nT)}\right)$$ (dependent on n)

This implies again as usual a diffusive process

$$\langle x(n)^2 \rangle = \left(\sqrt{2DTn}\right)^2 \propto \left(\sqrt{n}\right)^2$$ i.e the position after n-steps.

Consider now a transformed /scaled variable:

$$u_n \equiv \frac{x_n}{\sqrt{n}}$$

Then: the distribution is gaussian and independent on n

we rewrite:

$$p(x_n)\,dx = p(u_n)\cdot du_n = p(u_n)\cdot\left(\frac{dx}{\sqrt{n}}\right)$$

$$\hookrightarrow \quad p(u_n) = \sqrt{n}\cdot p(x_n)$$

$$\boxed{p(u_n) = \sqrt{n}\cdot\frac{1}{\sqrt{4\pi D(Tn)}}\, e^{-\left(\frac{u_n^2}{4DT}\right)}}$$ idependent on n!

$$\lim_{n\to\infty} \text{Prob}\{u_\alpha < u_n < u_\beta\} = \frac{1}{\sqrt{2\pi\sigma}} \int_{u_\alpha}^{u_\beta} \cdot\, e^{-u^2/2\sigma^2}$$ entirely independent on n.

→ that is on large scales of trajectories the random walks resemble ordinary Brownian motion. (Central limit Theorem)

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Lecture 7
LEVY Flights & anomalous diffusion

Levy flights are random walks where the central limit Theorem (CLT) does not apply.

Example $\boxed{P(y) = \frac{2b}{\pi(y^2+b^2)}}$ $y \geq 0$ CAUCHY DISTRIBUTION

Probability density function (PDF)

HEAVY TAILED DISTRIBUTION

Assume that transition probabilities obey:

$$P(x_1,n \mid x_2,n+1) = \begin{cases} P(y) & \text{for } y = x_2-x_1 > 0 \\ 0 & y < 0 \end{cases}$$

This implies (Chapman-Kolmogorov) Eqs.

$$P(x,n) = \int dx_{n-1} \cdots \int dx_1 P(x_1) \cdot P(x_2-x_1) \cdot \ldots \cdot P(x-x_{n-1})$$

using Laplace transform

$$\tilde{P}(s) = \int_0^\infty dy\, e^{-sy} P(y)$$

$$\tilde{P}(s,n) = \int_0^\infty dx\, e^{-sx} P(x,n) = [\tilde{P}(s)]^n$$

Assume next

$$\boxed{P(y) \simeq \frac{b}{y^{1+\mu}} \quad \text{for } y \to \infty}$$

For $0 < \mu \leq 2 \Rightarrow \langle y^2 \rangle = \int_0^\infty P(y) y^2 dy$ divergent

since for $\mu < 2$ i.e $\underline{(y^2 P(y)) \simeq y^{-(\mu-1)}}$ for $y \to \infty$

For $0 < \mu \leq 1$ $\langle y \rangle = \int_0^\infty P(y) y\, dy$ is divergent

since: $y \cdot P(y) \simeq (y^{-\mu})$ for $y \to \infty$.

Thus the Cauchy PDF leads to divergent 2nd and also divergent first moment.

Limiting distribution (Levy distribution)

Is there also a scaled variabel of position (y) such that the PDF becomes independent of n as $n \to \infty$? (and $X = \sum_1^n x_n$)

The CLTheorem does not apply any more hence one needs the Levy Khinchin Theorem.

If $\boxed{u_n = \frac{X_n}{n^{1/\beta}} \quad \text{then} \quad \lim_{n\to\infty} W_{\mu,b}(X,n) = W_{\mu,b}(X)}$ for $0<\mu<1$

Lecture 7
Levy Flights

This implies that $\boxed{x(n) \underset{n\to\infty}{\simeq} n^{1/\mu}}$

Hence for $0<\mu<1$ the growth is faster than diffusive $(x(n) \simeq \sqrt{n} = n^{1/2})$ and also faster than ballistic $(x(n) \propto n)$. $\Rightarrow$ SUPERDIFFUSIVE

The Levy distribution has the form: $(\mu = 1/2)$

$$\boxed{L_{1/2,b}(n) = \frac{b}{n^{3/2}} e^{-\pi b^2/n}}$$

$n \geq 0$ elsewhere zero.
SCALING FUNCTION

Where do levy statistics occur?

- frequency of geomagnetic reversal
- atomic laser cooling [$\to$ C. Cohen-Tannoudji, Nobel Prize Physics 1997)
- human travel
- length of path by photon in terbid media
- optimum search strategy

Analysis of cases $1 \leq \mu \leq 2$ yields:

$$\boxed{\begin{array}{ll} x(n) \simeq n \ln(n) & \mu = 1 \\ x(n) - n\langle y\rangle \simeq n^{1/\mu} & 1<\mu<2 \\ x(n) - n\langle y\rangle \simeq \sqrt{n \ln(n)} & \mu = 2 \end{array}}$$

(i.e. $u_n = \frac{x(n) - n\langle y\rangle}{n^{1/\mu}}$ gives stable process)

LEVY Flights "how rare events dominate statistics"

Let us calculate the most likely maximum increment in a series of n-steps:

Probability to observe increment $y \geq \bar{y}$, $P(\bar{y}, n)$

$$\boxed{Q(\bar{y}) = \int_{\bar{y}}^{\infty} dy\, P(y)}$$ Probability to observe increment $y > \bar{y}$

What equation does $Q(\bar{y})$ obey?

$$\boxed{Q(\bar{y}, n) = n \cdot Q(\bar{y}) (1 - Q(\bar{y}))^{n-1}}$$

since $[1-Q(\bar{y})]^{n-1}$ Probability to have $y < \bar{y}$. ($n-1$ times)

n: random occurring largest step in series.

Most likely probable value $\bar{y}_n$ is given by $\boxed{\frac{dQ(\bar{y},n)}{d\bar{y}} = 0}$

$$\Leftrightarrow \frac{d}{d\bar{y}} Q(\bar{y}, n) = n\left(\frac{d}{d\bar{y}} Q(\bar{y})\right)(1-Q(\bar{y}))^{n-1} + n Q(\bar{y}) \frac{dQ(\bar{y})}{d\bar{y}} (1-Q(\bar{y}))^{n-2} \times (n-1)$$

$$= n \frac{dQ}{d\bar{y}} \cdot (1-Q(\bar{y}))^{n-2} (1 - nQ(\bar{y})) \times \underbrace{\frac{dQ}{d\bar{y}}}_{\hat{=} \frac{d}{d\bar{y}} \int_{\bar{y}}^{\infty} P(y) dy = P(\bar{y})} \stackrel{!}{=} 0$$

Thus: $(1 - nQ(\bar{y})) = 0$

which yields:

$$Q(\bar{y}) = \int_{\bar{y}}^{\infty} dy\, P(y) = \frac{1}{n}$$

Let us consider Brownian Motion:

$$\frac{1}{n} = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\bar{y}_n}^{\infty} e^{-y^2/2\sigma^2}\, dy \underset{n\to\infty}{\simeq} \sqrt{\frac{2}{\pi}}\, \sigma\, e^{-\bar{y}^2/2\sigma^2}$$

Hence:

$$\boxed{\bar{y}_n \underset{n\to\infty}{\simeq} \sigma\sqrt{2\ln(n)}}$$

This implies the maximum likely stepsize is growing much slower than the increment itself. $y_n \propto \sqrt{n}$

$$\lim_{n\to\infty} \frac{\bar{y}_n}{X(n)} = \lim_{n\to\infty} \frac{\sigma\sqrt{2\ln(n)}}{\sqrt{n}} = 0$$

⇒ The maximum likely stepsizes do not contribute appreciably to increment $X(n)$.

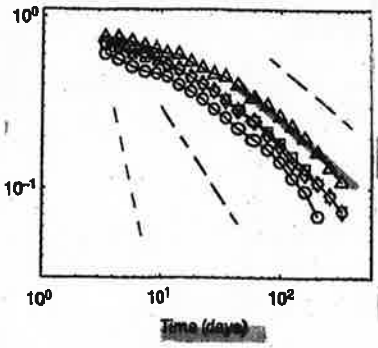
Now consider $P(y) \simeq \frac{b}{y^{1+\mu}}$ (anomalous diffusion)

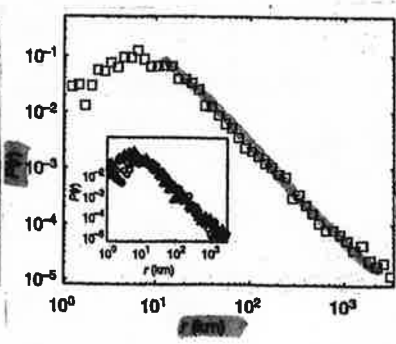
$$\frac{1}{n} \cong \int_{\bar{y}_n}^{\infty} \frac{b}{y^{1+\mu}}\, dy = \frac{b}{\mu} \frac{1}{\bar{y}_n^{\mu}} \quad \text{for } \mu > 0$$

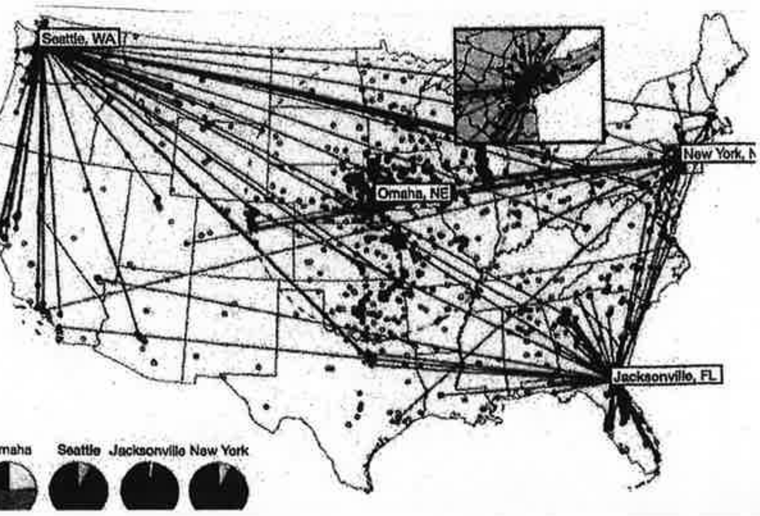
$$\Rightarrow \quad \boxed{\bar{y}_n \underset{n\to\infty}{\cong} n^{1/\mu}}$$

This implies however that the maximum increment is of the same order as $X(n) \simeq n^{1/\mu}$ for $1 < \mu < 2$.

Thus some rare events dominate statistics. These are called LEVY FLIGHTS.







LECTURE 7

Levy Flights

reference: T. Geisel, Nature 2006

Continuous time random walks (CTRW)

Example: two Prob. density functions $f(\Delta x)$ and $\Phi(\Delta t)$ that are statistically independent

$$P(y, \Delta t) = f(y)\,\Phi(\Delta t)$$

Thus

$$X_n = \sum_{n=1}^{N} y_n$$

$$T_n = \sum_{n=1}^{N} \Delta t_n$$

Ordinary diffusion:

f: Gaussian distribution ; $\Phi(\Delta t)$: exponential distribution

$$\Rightarrow W(x,t) \sim \frac{1}{\sqrt{t}} e^{-x^2/Dt} \qquad X(t) \sim \sqrt{t}$$

Levy flight

$f(y) \simeq \frac{1}{y^{(\beta+1)}}$ $\quad \Phi(\Delta t)$ finite expectation value $\quad T = \int_0^\infty \Phi(\Delta t)\,\Delta t\,dt$

$$W(x,t) \sim \frac{1}{t^{1/\beta}} L_\beta\left(\frac{x}{t^{1/\beta}}\right)$$

L is stable process of index β.

$$\Leftrightarrow X(t) \sim t^{1/\beta}$$

Fractional Brownian Motion (SUB-DIFFUSION) $\quad 0<\alpha<1$

$\Delta\Phi(\Delta t) \sim \frac{1}{\Delta t^{(1+\alpha)}}$ and $f(y)$ is Gaussian PDF.

$$W(x,t) \sim \frac{1}{t^{\alpha/2}} L_\alpha\left(\frac{x}{t^{\alpha/2}}\right)$$

$$X(t) \sim t^{\alpha/2}$$

Ambivalent processes

$f(y) \cong \frac{1}{y^{1+\beta}}$ $\quad 0<\beta<2$ $\quad \Phi(\Delta t) \cong \frac{1}{\Delta t^{1+\alpha}}$ $\quad$ (for $\Delta t, y \to \infty$)

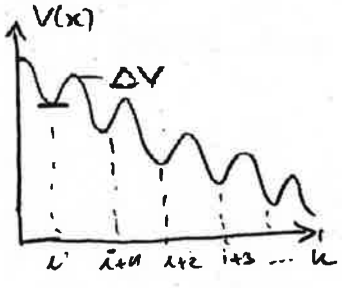
$X(t) \sim t^{\alpha/\beta}$

- $\Rightarrow \frac{1}{2}\alpha = \beta$ ordinary diffusion
- $\Rightarrow \frac{1}{2}\alpha > \beta$ SUPER-DIFFUSION
- $\frac{1}{2}\alpha < \beta$ SUB-DIFFUSION

LECTURE 7

Levy flights

Example Arrenius cascade



Heights ΔV distributed according to $f(u) = \frac{1}{E_0} e^{-\Delta V/E_0}$ Distribution of heights

$T_i = T_0 e^{-\Delta V/k_B T}$

$T(n) = \sum_{i=1}^{n} T_i$

Using the expression: $P(T)\,dT = P(V)\,dV$

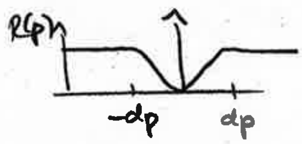
$\Leftrightarrow p(T) = P(V(T)) \frac{d}{dT} V(T) \wedge V(T) = k_B T \ln(T/T_0)$

$\Leftrightarrow \underline{P(T) = \mu \cdot \frac{T_0^{\mu}}{T^{1+\mu}}}$ and $\mu = \frac{k_B T}{E_0}$

Thus for $k_B T < 2E_0$ the statistics is dominated by rare events, long tailed distribution.

$T(n) \sim n^{1/\mu} = n^{E_0/k_B T}$ grows much faster than linear.

→ This is exploited in sub-Doppler laser cooling to bring atoms to rest ($\rightarrow R(p) \propto \delta . p^2$ R=const otherwise)



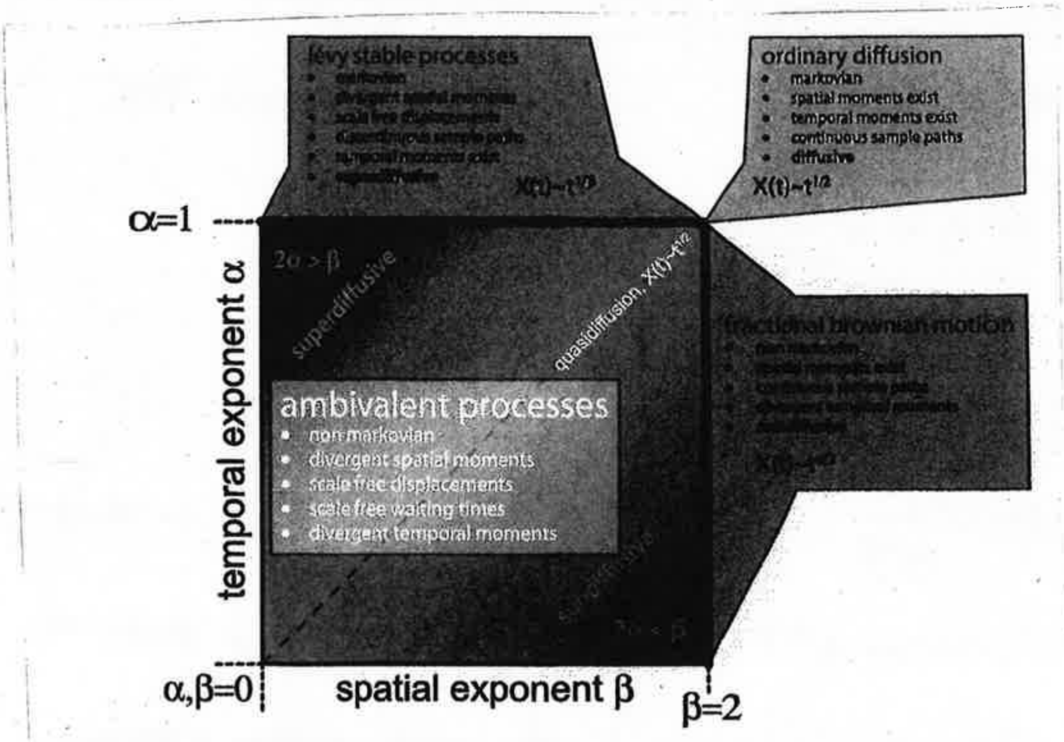


Figure 1.2: The asymptotic universality classes of continuous time random walks defined in the text as a function of the universality exponents $0 < \alpha < 1$ and $0 < \beta < 2$. Lévy flights, fractional Brownian motion as well as ordinary diffusion are limiting cases of the more general class of ambivalent processes.

 (2014)

LECTURE 8

Master equations

Example:
Death and Birth processes

Suggested Reading:

(1) Chapter 11 "Stochastic Methods" C. Gardiner
(2) Chapter Poly_copy P. Martin

Transition rates: Assume we have a system

$$W(u_1,u_2) = \lim_{t\to 0}\left(\frac{P(u_1,0\,|\,u_2,t)}{t}\right) = \frac{\partial}{\partial t}\left[P(u_1,0\,|\,u_2,t)\right]$$

Transition probability

let use assume the transition probability:

$$P(u_1,0\,|\,u_2,t) = W(u_1,u_2)\cdot t + O(t^2)$$

Derive an Equation of motion for P:

$$P(u_1|u_2,\Delta t) = \begin{cases} u_1 = u_2 & \text{prob stay in same state during } \Delta t \\ u_1 \neq u_2 & \text{prob. to arrive in state } u_2 \text{ during } \Delta t \end{cases}$$

Introduce $\boxed{a(u_1) = \sum_{u_1\neq u_2} W(u_1,u_2)}$ Transition rate out of state u_1

Derive equation of motion of $P(u_1|u_2,\Delta t)$

$$P(u_1|u_2,\Delta t) = \underbrace{(1-a(u_1)\Delta t)\,\delta_{u_1,u_2}}_{\text{Probability to stay in state } u_1=u_2} + \underbrace{(1-\delta_{u_1,u_2})\,W(u_1|u_2)\Delta t}_{\text{Probability to exit if } u_1 = u_2}$$

For a Markov process the Chapman Kolmogorov equations give

$$P(u_1|u_3,t+\Delta t) = \sum_{u_2\in S} P(u_1|u_2,t)\,\underbrace{P(u_2,t\,|\,u_3,t+\Delta t)}_{=P(u_2|u_3+\Delta t)}$$

for a homogeneous process.

$$= \sum_{u_2} P(u_1|u_2,t)\left[(1-a(u_2)\Delta t)\,\delta_{u_2,u_3} + (1-\delta_{u_2,u_3})\,W(u_2|u_3)\Delta t\right]$$

$$P(u_1|u_3,t+\Delta t) = P(u_1|u_3,t) - P(u_1|u_3,t)\,a(u_3)\,\Delta t + \sum_{u_2} P(u_1|u_2,t)\,W(u_2|u_3)\,\Delta t \times (1-\delta_{u_2,u_3})$$

$$= P(u_1|u_3,t) - P(u_1|u_3,t)\,a(u_3)\,\Delta t + \sum_{\substack{u_2 \\ u_2 \neq u_3}} P(u_1|u_2,t)\,W(u_2|u_3)\,\Delta t$$

$$= P(u_1|u_3,t) - P(u_1|u_3,t)\underbrace{\Big[\sum_{\substack{u_2\in S \\ u_2\neq u_3}} W(u_3|u_2)\,\Delta t\Big]}_{\hat{=}\, a(u_3)} + \sum_{\substack{u_2\in S \\ u_2\neq u_3}} P(u_1|u_2,t)\,W(u_2|u_3)\cdot\Delta t$$

$$\Leftrightarrow \lim_{\Delta t\to 0}\left(\frac{P(u_1|u_3,t+\Delta t)-P(u_1|u_3,t)}{\Delta t}\right) = \sum_{\substack{u_2\in S \\ u_2\neq u_3}} P(u_1|u_2,t)\,W(u_2|u_3) - P(u_1|u_3,t)\,W(u_3|u_2)$$

Note that for $u_2 = u_3$:

$$P(u_1|u_2,t)\,W(u_2,u_2) - P(u_1|u_2,t)\,W(u_2|u_2) = 0$$

Mesoscopic Master Equation

$$\boxed{\frac{\partial}{\partial t} P(u,t) = \sum_{u'\in S} P(u',t)\,W(u'|u) - P(u,t)\,W(u|u')}$$

For a stationary process:

$$\begin{tikzcd} u \arrow[r, bend left, "W(u|u')"] & u' \arrow[l, bend left, "W(u'|u)"] \end{tikzcd}$$

$$\boxed{\sum_{u'\in S} P^S(u')\,W(u'|u) - P^S(u)\,W(u|u') = 0}$$